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EXTENSION OF POLYNOMIAL MAPPINGS WITH A GIVEN ŁOJASIEWICZ EXPONENT

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Abstract. Let $V \subset \mathbb{C}^n$ be an affine subspace. We prove that for a polynomial mapping $f : V \to \mathbb{C}^m, n \leq m$, there is an extension $F : \mathbb{C}^m \to \mathbb{C}^m$ with the same Łojasiewicz exponent at infinity.

Let V be an infinite algebraic subset of $\mathbb{C}^n$ and let $f : V \to \mathbb{C}^m$ be a polynomial mapping. By the Łojasiewicz exponent at infinity (or the global Łojasiewicz exponent) of f, we mean the number $\mathcal{L}_\infty(f) = \sup\{\nu \in \mathbb{R} : \exists A, B > 0\ \forall z \in V\ \ |z| > B \Rightarrow A\,|z|^\nu \leq |f(z)|\}$. The number $\mathcal{L}_\infty(f)$ does not depend on the choice of norms and on linear change of coordinates (see e.g. [**1**]). Thus in the sequel we will use the maximum norm.

In this note we prove the following:

Theorem 1. *For every affine subspace $V \subset \mathbb{C}^n$ and for every polynomial mapping $f : V \to \mathbb{C}^m$, with $n \leq m$, there exists a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ with $F|_V = f$ and $\mathcal{L}_\infty(F) = \mathcal{L}_\infty(f)$.*

The above result is the very first step in a research into the following question: *For what kind of sets $V \subset \mathbb{C}^n$ and what kind of mappings $f : V \to \mathbb{C}^m, n \leq m$, does there exist a polynomial extension $F : \mathbb{C}^n \to \mathbb{C}^m$ of f with $\mathcal{L}_\infty(F) = \mathcal{L}_\infty(f)$?* or into a more general one: *What can we say about* $\max\{\mathcal{L}_\infty(F) : F|_V = f\}$?

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To prove Theorem 1, first notice that for any algebraic subset Z of $\mathbb{C}^m$ and any projection $\pi : Z \to \mathbb{C}^k$ there is $\mathcal{L}_\infty(\pi) \leq 1$. But, by Rudin–Sadullayev theorem (see e.g. [**2**], VII.7.4), it is easy to see that the following is true.

Proposition 2. (see also [**3**], Theorem 2.1) *Let Z be an algebraic subset of $\mathbb{C}^m$ of pure dimension k. Then there exists a linear change of coordinates in $\mathbb{C}^m$ such that for the projection $\pi : Z \to \mathbb{C}^k \times \{0\} \subset \mathbb{C}^m$ there is $\mathcal{L}_\infty(\pi) = 1$.*

Thus, it is natural to say that a projection is a *Sadullayev projection* iff its global Łojasiewicz exponent is equal to one (has the maximum possible value).

Proof of Theorem 1. Without loss of generality we may assume that $\dim V = k < n$ and $V = \mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$. We may also assume that f is a non constant mapping, and then that $l = \dim \overline{f(V)} > 0$. By Proposition 2, we may also assume that the projection $p : \overline{f(V)} \to \mathbb{C}^l \times \{0\} \subset \mathbb{C}^l \times \mathbb{C}^{m-l}$ is a Sadullayev projection. Thus $\mathcal{L}_\infty((f_1, \ldots, f_l)) = \mathcal{L}_\infty(p \circ f) = \mathcal{L}_\infty(f)$. Since, also, $\mathcal{L}_\infty(p \circ f) \leq \mathcal{L}_\infty(f') \leq \mathcal{L}_\infty(f)$, where $f' = (f_1, \ldots, f_k) : V \to \mathbb{C}^k$ (because $l \leq k$ and we use the maximum norm), then $\mathcal{L}_\infty(f') = \mathcal{L}_\infty(f)$. In the sequel we will use the notation $(x, y) = (x_1, \ldots, x_k, y_{k+1}, \ldots, y_n)$ for points in $\mathbb{C}^k \times \mathbb{C}^{n-k} \cong \mathbb{C}^n$. Let us take a $d \in \mathbb{N} \setminus \{0\}$ such that $\max\{\mathcal{L}_\infty(f), \deg f_{k+1}, \ldots, \deg f_n\} < d$ and put

$$\widetilde{F} : \mathbb{C}^n \ni (x, y) \mapsto (f_1(x), \ldots, f_k(x), f_{k+1}(x) + y_{k+1}^d, \ldots, f_n(x) + y_n^d) \in \mathbb{C}^n,$$
$$F : \mathbb{C}^n \ni (x, y) \mapsto (\widetilde{F}(x, y), f_{n+1}(x), \ldots, f_m(x)) \in \mathbb{C}^m.$$

Obviously, there is $F|_V = \widetilde{F}|_V = f$. If $\nu \leq \mathcal{L}_\infty(f) = \mathcal{L}_\infty(f')$, then there exist $A_1 > 0, B_1 > 0$, such that $|x| > B_1 \Rightarrow A_1 |x|^\nu \leq |f'(x)|$. Let $C, D > 0$ and $B_2 > 0$ be such that

$$|x| > D \Rightarrow |f_i(x)| \leq C |x|^{\deg f_i}, \quad \frac{C}{B_2^{d - \deg f_i}} < \frac{1}{2}, \qquad \text{for } i = k+1, \ldots, n.$$

Put $B = \max\{B_1, B_2, D, 1\}$ and $A = \min\{A_1, \frac{1}{2}\}$. Take an arbitrary $(x, y) \in \mathbb{C}^n$ with $|(x, y)| > B$. If $|(x, y)| = |x|$, then

$$\left|\widetilde{F}(x, y)\right| \geq |f'(x)| \geq A_1 |x|^\nu = A_1 |(x, y)|^\nu \geq A |(x, y)|^\nu.$$

Now assume that $|(x, y)| = |y|$ and choose an $i \in \{k+1, \ldots, n\}$ such that $|(x, y)| = |y| = |y_i|$. Then

$$\begin{aligned}
\left|\widetilde{F}(x, y)\right| &\geq \left|f_i(x) + y_i^d\right| \geq \left|y_i^d\right| - |f_i(x)| \geq |y_i|^d - C |x|^{\deg f_i} \\
&\geq |(x, y)|^d - C |(x, y)|^{\deg f_i} = \left(1 - \frac{C}{|(x, y)|^{d - \deg f_i}}\right) |(x, y)|^d \\
&\geq A |(x, y)|^d \geq A |(x, y)|^\nu.
\end{aligned}$$

This proves the inequality $\mathcal{L}_\infty(\widetilde{F}) \geq \mathcal{L}_\infty(f)$, and since $\mathcal{L}_\infty(F) \geq \mathcal{L}_\infty(\widetilde{F})$ (we use the maximum norm), the inequality $\mathcal{L}_\infty(F) \geq \mathcal{L}_\infty(f)$ holds. The opposite one follows directly from the definitions of $\mathcal{L}_\infty(F)$ and $\mathcal{L}_\infty(f)$. $\square$

References

1. Chądzyński J., Krasiński T., *A set on which the Łojasiewicz exponent at infinity is attained,* Ann. Polon. Math., **67** (1997), 191–197.
2. Łojasiewicz S., *Introduction to complex analytic geometry*, Birkhäuser Verlag, 1991.
3. Spodzieja S., *The Łojasiewicz exponent at infinity for overdetermined polynomial mappings*, Ann. Polon. Math., **78** (2002), 1–10.

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